

STOCHASTIC BANACH PRINCIPLE IN OPERATOR ALGEBRAS

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ABSTRACT. Classical Banach principle is an essential tool for the investigation of the ergodic properties of Cesaro subsequences.

The aim of this work is to extend Banach principle to the case of the stochastic convergence in the operator algebras.

We start by establishing a sufficient condition for the stochastic convergence (stochastic Banach principle). Then we formulate stochastic convergence for the bounded Besicovitch sequences, and, as consequence for uniform subsequences.

1. INTRODUCTION AND PRELIMINARIES

In this paper we establish a Stochastic Banach Principle. The Banach Principle is one of the most useful tools in "classical" point-wise ergodic theory. The Banach principle was used to give an alternative proof of the Birkhoff- Khinchin individual ergodic theorem. Typical applications of the Banach Principle are Sato's theorem for uniform subsequences [17] and individual ergodic theorem for the Besicovitch Bounded sequences [15] . Non-commutative analogs for the (double side) almost everywhere convergence may be found in papers [8], [2].

In this paper we establish a Banach Principle for convergence in measure (Stochastic Banach Principle, Theorem 3.2.3). We reformulate the theorem in a form convenient for applications (Theorem 3.2.4). Based on the principle we give a simplified proof of the stochastic ergodic theorem (compare with [9]). We establish stochastic convergence for Sato's uniform subsequences (Theorem 3.6) and a stochastic ergodic theorem for the Besicovitch Bounded sequences (Theorem 3.7).

Note that these results are new even in the commutative case.

Throughout the paper we denote by M a von Neumann algebra with semi-finite normal faithful trace τ acting on Hilbert space \mathfrak{H} . Denote by $P(M)$ the set of all orthogonal projections in M .

Recall the following definitions (combined from the papers by Segal [16], Nelson [14], Yeadon [20], Fack and Kosaki [6]):

Definition 1.1. A densely defined closed operator x affiliated with von Neumann algebra M is called **(τ) measurable** if for every $\epsilon > 0$ there exists projection $e \in P(M)$ with $\tau(\mathbb{I} - e) < \epsilon$ such that $e(\mathfrak{H}) \subset \mathfrak{D}(x)$, where $\mathfrak{D}(x)$ is a domain of x .

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Space of all (τ) **measurable operators affiliated with M** is denoted by $S(M)$.

For convenience for a self-adjoint $x \in S(M)$ we denote by $\{x > t\}$ the **spectral projection of x corresponding to the interval $(t, \infty]$** .

Definition 1.2. Sequence $\{x_n\}_{n=1}^\infty$ **converges to 0 in measure (stochastically)** if for every $\epsilon > 0$ and $\delta > 0$ there exists an integer N_0 and a set of projections $\{e_n\}_{n \geq N_0} \subset P(M)$ such that $\|x_n e_n\|_\infty < \epsilon$ and $\tau(\mathbb{I} - e_n) < \delta$ for $n \geq N_0$.

Remark 1.1. We will use terms **converges in measure** and **converges stochastically** interchangeably.

Definition 1.3. Let x be a measurable operator from $S(M)$ and $t > 0$. The **t -th singular number of x** is defined as

$$(1) \quad \mu_t(x) = \inf\{\|xe\| \mid e \text{ is a projection in } P(M) \text{ with } \tau(\mathbb{I} - e) < t\}.$$

Remark 1.2. Note that **measure topology** is defined in Fack and Kosaki's [6] as linear topology with fundamental system of neighborhoods around 0 given by $V(\epsilon, \delta) = \{x \in S(M) \text{ such that there exists a projection } e(x, \epsilon, \delta) \text{ with } \|xe\| < \epsilon \text{ and } \tau(\mathbb{I} - e) < \delta\}$.

Definition 1.4. Denote by $\lambda_t(x)$ the **distribution function of x** defined as

$$(2) \quad \lambda_t(x) = \tau(E_{(t, \infty)}(|x|)), \quad t \geq 0,$$

here $E_{(t, \infty)}(|x|)$ is a spectral projection of x corresponding to interval (t, ∞) .

Remark 1.3. For the measurable operator x , we have $\lambda_t(x) < \infty$ for large enough t and $\lim_{t \rightarrow \infty} \lambda_t(x) = 0$. Moreover, the map $\mathbb{R} \ni t \rightarrow \lambda_t(x)$, is non-increasing and continuous from the right (because τ is normal and $\{|x| > t_n\} \uparrow \{|x| > t\}$ (and hence in strong operator topology) as $t_n \downarrow t$). The distribution $\lambda_t(x)$ is a non-commutative analogue of the distribution function in classical analysis, (see. [6] p. 272 or [18]).

We would need the following statement about properties of the $\mu_t(x)$ (see for example proposition 2.4 [20], or lemma 2.5 [6]):

Lemma 1.1. *Let $x, y \in S(M)$ be measurable operators.*

- i) *Map $\mathbb{R} \ni t \rightarrow \mu_t(x)$ is non-decreasing and continuous from the right.*
Moreover $\lim_{t \downarrow 0} \mu_t(x) = \|x\|_\infty \in [0, \infty]$,
- ii) *$\mu_t(x) = \mu_t(|x|) = \mu_t(x^*)$ and $\mu_t(\alpha x) = |\alpha| \mu_t(x)$ for $\alpha \in \mathbb{C}$, $t > 0$,*
- iii) *$\mu_t(x) \leq \mu_t(y)$ for $0 \leq x \leq y$, $t > 0$,*
- iv) *$\mu_{t+s}(x+y) \leq \mu_t(x) + \mu_s(y)$ for $t, s > 0$,*
- v) *$\mu_t(yxz) \leq \|y\|_\infty \|z\|_\infty \mu_t(x)$, for $y, z \in M$, $t > 0$,*
- vi) *$\mu_{t+s}(yx) \leq \mu_t(x) \mu_s(y)$ for $t, s > 0$.*

2. STOCHASTIC BANACH PRINCIPLE

We start the section with the description of some conditions equivalent to stochastic convergence (cmp. with lemma 3.1 [6]).

Lemma 2.1. *Let M , τ be as before. Consider following conditions:*

- i) *Sequence $\{x_n\}_{n=1}^\infty$ converges to 0 in measure,*
- ii) *For every $\epsilon > 0, \delta > 0$ there exist a positive real $0 < \delta' < \delta$ and an integer N_0 such that for $n \geq N_0$*

$$\mu_{\delta'}(x_n) < \epsilon,$$

- iii) For every $\epsilon > 0, \delta > 0, p \in P(M)$ with $\tau(p) < \infty$ there exists an integer N_0 and a sequence of projections $\{e'_n\}_{n \geq N_0} \subset P(M)$, $e'_n \leq p$ such that

$$\|x_n e'_n\|_\infty < \epsilon \text{ and } \tau(p - e'_n) < \delta \text{ for } n \geq N_0.$$

The following relations take place: $i) \Leftrightarrow ii) \Rightarrow iii)$. If τ is finite then $iii) \Rightarrow i)$.

Proof. Implication $ii) \Rightarrow i)$ follows from the fact that condition $\mu_{\delta'}(x_n) < \epsilon$ implies for the sequence of projections $\{e_n\}_{n=1}^\infty$, holds $\|x_n e_n\| \leq 2\epsilon$ and $\tau(\mathbb{I} - e_n) \leq \delta'$. Implication $i) \Rightarrow ii)$ follows from the definition of measure convergence 1.2.

Implication $i), ii) \Rightarrow iii)$ follows from the inequality

$\tau(p - p \wedge q) = \tau(p \vee q - q) \leq \tau(\mathbb{I} - q)$, hence sequence $\{e'_n = e_n \wedge p\}_{n=1}^\infty$ satisfies iii) (here projections e_n are defined in the proof $ii) \Rightarrow i)$).

The case when τ is finite follows immediately since $\tau(\mathbb{I}) < \infty$. \square

We need the following technical statement which is interesting by itself:

Lemma 2.2. *Let x, y be self-adjoint measurable operators from $S(M)$, t, s be positive real. Then*

$$(3) \quad \lambda_{t+s}(x + y) \leq \lambda_t(x) + \lambda_s(y)$$

Proof. Indeed,

$$\begin{aligned} (4) \quad & \| (x + y) |(\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})\| = \\ & \| (x + y) (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| \leq \\ & \| x (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| + \| y (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| = \\ & = \| |x| (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| + \| |y| (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \| \leq \\ & \leq \| |x| (\mathbb{I} - \{|x| > t\}) \| + \| |y| (\mathbb{I} - \{|y| > s\}) \| \leq \\ & t + s. \end{aligned}$$

Here the first and the second equality follows from the equality $\| |z| u_z^* u_z |z| \| = \| |z|^2 \| = \| z^* z \|$, where $z \in M_h$, u_z is a partial isometry from M such that $z = u_z |z|$, and

$$(5) \quad u_z^* u_z = l(z), \quad u_z u_z^* = r(z),$$

where $l(z)(r(z))$ is a left (right) support of z . Inequality 4 means that

$$(6) \quad \mu_{\lambda_t(x) + \lambda_s(y)}(x + y) \leq t + s.$$

Let ξ be a vector from Hilbert space \mathfrak{H} and suppose that

$$(7) \quad \xi \in \{|x + y| > s + t\} \mathfrak{H} \cap (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \mathfrak{H}.$$

Then

$$(8) \quad ((t + s)\|\xi\|)^2 < (|x + y|\xi, |x + y|\xi) = ((x + y)\xi, (x + y)\xi) \leq ((t + s)\|\xi\|)^2,$$

here the first inequality follows from inclusion $\xi \in \{|x + y| > s + t\} \mathfrak{H}$, the equality follows from the spectral decomposition 5, the second inequality follows from inclusion $\xi \in (\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\}) \mathfrak{H}$.

Inequality 8 implies that $\|\xi\| = 0$ or, in other words,

$$\{|x + y| > s + t\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = 0.$$

Hence,

$$\begin{aligned}
 \{|x+y| > t+s\} &= \{|x+y| > t+s\} - \\
 &\{|x+y| > t+s\} \wedge ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \sim \\
 (9) \quad &\sim \{|x+y| > t+s\} \vee ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) - \\
 &((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) \leq \\
 &\leq \mathbb{I} - ((\mathbb{I} - \{|x| > t\}) \wedge (\mathbb{I} - \{|y| > s\})) = \{|x| > t\} \vee \{|y| > s\}.
 \end{aligned}$$

Here \sim means projection equivalence. Since trace τ is invariant on equivalent projections,

$$(10) \quad \tau(\{|x+y| > t+s\}) \leq \tau(\{|x| > t\} \vee \{|y| > s\}) \leq \tau(\{|x| > t\}) + \tau(\{|y| > s\})$$

and, hence, the inequality (3) takes place. \square

Theorem 2.3. *Let $(B, \|\cdot\|)$ be a Banach space. Let $\Sigma = \{A_n, n \in \mathbb{N}\}$ be a set of linear operators $A_n : B \rightarrow S(M)$.*

- i) *Suppose that there exists a function $C(\lambda) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$, and such that*

$$(11) \quad \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda)$$

holds for every $b \in B$, $\lambda \in \mathbb{R}_+$.

Then the subset \bar{B} of B where $A_n(b)$ converges in measure (stochastically) is closed in B .

- ii) *Conversely, if A_n is a set of continuous in measure maps from B into $S(M)$ and for each $b \in B, \lambda \in \mathbb{R}_+$*

$$(12) \quad \lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0,$$

then there exists a function $C(\lambda) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\lim_{\lambda \rightarrow \infty} C(\lambda) = 0$, and

$$(13) \quad \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|\}) \leq C(\lambda)$$

Part i) of the theorem 2.3 means that under the condition of the linear uniform boundness (11), the set of the stochastic convergence is closed.

Part ii) of the theorem 2.3 means that if the set of uniform boundness is closed, then linear uniform boundness takes place.

Note that even the condition in the part ii) looks more restrictive, it is similar in nature to the condition of part i), since we can restrict everything to the closed linear subspace B_1 of Banach space B (B_1 is also Banach space).

Proof. Part i) We first show that condition 11 implies continuity of the set Σ of operators. Let $B \supset \{b_k\}_{k=1}^\infty$ be a sequence in B converging to $b \in B$. Then for $\lambda, \epsilon \in \mathbb{R}_+$ with $2\lambda \sup_{k \geq n} \|b - b_k\| < \epsilon$

$$\begin{aligned}
 (14) \quad \tau(\{|A_n(b_k) - A_n(b)| > \epsilon\}) &\leq \tau(\{|A_n(b_k - b)| > \lambda \|b_k - b\|\}) \leq \\
 &C(\lambda \|b_k - b\|^{-1}) \xrightarrow{b_k \rightarrow b} 0,
 \end{aligned}$$

for $k \geq n$, and, hence A_n continuous. Note that the inequality follows from the fact that right part of 11 does not depend on the norm of b .

There exists a subsequence b_{k_j} of b_k such that sequence $x_j = \lim_{n \rightarrow \infty} A_n(b_{k_j})$ converges stochastically. To show this we choose a sequence of $\{k'_i\}_{i=1}^\infty$ base on the inequality 11 in such a manner that

$$(15) \quad \tau(\{|A_n(b_{k'_j} - b_{k'_{j+l}})| > 2^{-j}\}) \leq 2^{-j} \text{ for all } n$$

and

$$(16) \quad \tau(\{|A_n(b_{k'_j} - b)| > 2^{-j}\}) \leq 2^{-j} \text{ for all } n$$

This may be done since $b_n \xrightarrow{n \rightarrow \infty} b$, and $C(\lambda) \xrightarrow{\lambda \rightarrow \infty} 0$. It is sufficient to choose a sequence of $\{\lambda_j\}_{j=1}^\infty$ in such a way that $C(\lambda_j) < 2^{-j}$ and $\|b_{k_j} - b\| < \lambda_j^{-1} 2^{-2j}$.

Choose n_j in such a manner that for $N > n_j$ holds

$$(17) \quad \tau(\{|A_N(b_{k_j}) - x_j| > 2^{-j}\}) < 2^{-j}.$$

This is possible since $A_n(b_{k_j})$ converges stochastically to x_j .

Then for $j, i \in \mathbb{N}$ and $n > n_{i+j}$

$$(18) \quad \begin{aligned} \tau(\{|x_j - x_{j+i}| > 3 \cdot 2^{-j}\}) &= \tau(\{|(x_j - A_n(b_{k_j})) + \\ &\quad (A_n(b_{k_j}) - A_n(b_{k_{j+i}})) + (A_n(b_{k_{j+i}}) - x_{j+i})| > 3 \cdot 2^{-j}\}) \leq \\ &\tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\}) + \tau(\{|A_n(b_{k_j}) - A_n(b_{k_{j+i}})| > 2^{-j}\}) + \\ &\tau(\{|A_n(b_{k_{j+i}}) - x_{j+i}| > 2^{-(j+i)}\}) \leq 3 \cdot 2^{-j}. \end{aligned}$$

Here the first inequality follows from the 3.

Denote the stochastic limit of $\{x_j\}_{j=1}^\infty$ by x_0 . If necessary by taking a subsequence of $\{x_j\}$ and reindexing, we suppose that

$$(19) \quad \tau(\{|x_j - x_0| > 2^{-j}\}) \leq 2^{-j}.$$

Sequence $\{A_n(b)\}_{n=1}^\infty$ converges to x_0 stochastically. Indeed, for $n > n_j$ holds the following inequality holds

$$(20) \quad \begin{aligned} \tau(\{|A_n(b) - x_0| > 3 \cdot 2^{-j}\}) &= \tau(\{|(A_n(b) - A_n(b_{k_j})) + \\ &\quad (A_n(b_{k_j}) - x_j) + (x_j - x_0)| > 3 \cdot 2^{-j}\}) \leq \tau(\{|A_n(b) - A_n(b_{k_j})| > 2^{-j}\}) + \\ &\tau(\{|A_n(b_{k_j}) - x_j| > 2^{-j}\}) + \tau(\{|x_j - x_0| > 2^{-j}\}) \leq 3 \cdot 2^{-j}. \end{aligned}$$

Here the first inequality follows from 3, and the second inequality follows by noting that the first part follows from 16, the second part follows from 17 and choice of n , and the third part follows from 19.

Part i) is established.

Part ii) Suppose that for every $b \in B$ and $\lambda \in \mathbb{R}_+$ holds

$$(21) \quad \sup_n \tau(\{|A_n(b)| > \lambda\}) \xrightarrow{\lambda \rightarrow \infty} 0.$$

For fixed $\epsilon > 0$ and $\lambda \in \mathbb{N}$ define $B_\lambda = \{b \in B \mid \sup_n \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$. Then from 21 it follows that

$$(22) \quad B = \bigcup_{\lambda \in \mathbb{N}} B_\lambda$$

Let $B_{\lambda,k}$ be a set defined as $\{b \in B \mid \sup_{n \geq k} \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon\}$. Then

$$(23) \quad B_\lambda = \bigcap_{k \in \mathbb{N}} B_{\lambda,k}$$

Sets $B_{\lambda,k}$ are closed. Indeed, let $B_{\lambda,k} \supset \{b_j\}_{j=1}^\infty$ converges to $b \in B$. Then

$$(24) \quad \begin{aligned} \tau(\{|A_n(b)| > \lambda + \gamma\}) &= \tau(\{|A_n(b_k) - (A_n(b_j) - A_n(b))| > \lambda + \gamma\}) \leq \\ &\tau(\{|A_n(b_j)| > \lambda\}) + \tau(\{|(A_n(b_j) - A_n(b))| > \gamma\}) \leq \epsilon \end{aligned}$$

Here the first inequality follows from 3, the first estimate follows from the definition of $B_{\lambda,k}$ and the second estimate become valid for sufficiently large j , and follows from the free choice of b_j and continuity of A_n in measure.

Since $\lambda_t(x)$ is continuous from the right (1.3), then

$$(25) \quad \tau(\{|A_n(b)| > \lambda\}) = \lim_{m \rightarrow \infty} \tau(\{|A_n(b)| > \lambda + \gamma_m\}) \leq \epsilon,$$

where $\gamma_m \xrightarrow{m \rightarrow \infty} 0$. Hence, $b \in B_{\lambda,k}$, or $B_{\lambda,k}$ is closed. Set B_λ is closed as an intersection of closed sets (23).

It follows from the Baire category principle that there exists λ such that set B_λ has non empty interior. Let $B(b_0, r) = \{b \in B \mid \|b - b_0\| \leq r\}$ be contained in the B_λ .

Then

$$(26) \quad \tau(\{|A_n(b)| > \lambda\}) \leq \epsilon \text{ for every } b \in B(b_0, r).$$

Moreover, for $b = b_0 - r \cdot c \in B(b_0, r)$ with $c \in B$, $\|c\| \leq 1$, holds

$$(27) \quad \begin{aligned} \tau(\{|A_n(r \cdot c)| > 2 \cdot \lambda\}) &= \tau(\{|A_n(r \cdot c - b_0) + A_n(b_0)| > 2 \cdot \lambda\}) \leq \\ &\tau(\{|A_n(r \cdot c - b_0)| > \lambda\}) + \tau(\{|A_n(b_0)| > \lambda\}) \leq 2 \cdot \epsilon. \end{aligned}$$

Let $\gamma \geq 2 \cdot \lambda/r$. From 27 it follows that $\tau(\{|A_n(c)| > \gamma\}) \leq 2 \cdot \epsilon$, for every $c \in B$, $\|c\| \leq 1$.

Let $C(\gamma) = \sup_{c \in B, \|c\| \leq 1} \tau(\{|A_n(c)| > \gamma\}) \leq 2 \cdot \epsilon$. Free choice of ϵ implies that

$$(28) \quad \lim_{\gamma \rightarrow \infty} C(\gamma) = 0,$$

hence 11 is valid. \square

For the application of theorem 2.3 it is convenient to combine both parts i) and ii).

Theorem 2.4. *Let $(B, \|\cdot\|)$ be a Banach space. Let A_n be a set of continuous in measure linear maps from B into $S(M)$, let $\lambda \in R_+$, and for each $b \in B$ holds*

$$(29) \quad \lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda\}) = 0.$$

Then subset \tilde{B} of B where $A_n(b)$ converges in measure (stochastically) is closed in B .

Proof. Follows immediately from applying consecutively Theorem 2.3 part ii) then part i). \square

Let e be a projection in M , let M_e be von Neumann algebra consisting of operators of form exe , $x \in M$. If τ is a semifinite normal faithful trace on M then $\tau_e = \tau|_{M_e}$ is a semifinite (possibly finite) faithful normal trace on M_e . Indeed, tracial property, semifiniteness, normalness and faithfulness of τ_e follows directly from similar properties of τ . Space $S(M_e, \tau_e)$ is isomorphic to the $S(M, \tau)_e$ since both these spaces are closures of the $(M_{\tau\text{-finitesupport}})_e = (M_e)_{\tau_e\text{-finitesupport}}$.

Proposition 2.5. *Let B_n be a sequence of continuous in measure operators on $S(M, \tau)$. Let $e_i \in P(M)$, $i = 1, 2$, $\mathbb{I} = e_1 + e_2$ be projections in M . Suppose that relation $e_i(B_n(x)) = B_n(x_{e_i}) = (B_n(x))e_i$ holds for every $n \in \mathbb{N}$ and $x \in S(M, \tau)$, or, in other words, e_i commutes with B_n . Suppose also following relations hold*

$$(30) \quad \lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|B_n(x_{e_i})| > \lambda\}) = 0,$$

for $i = 1, 2$ and every $x \in S(M, \tau)$. Then the following equality is valid:

$$(31) \quad \lim_{\lambda \rightarrow \infty} \sup_{n \in \mathbb{N}} \tau(\{|B_n(x)| > \lambda\}) = 0.$$

Proof. The following relations are valid:

$$(32) \quad \tau(\{|B_n(x_{e_i})| > \lambda\}) = \tau(e_i\{|B_n(x)| > \lambda\}).$$

Indeed, since for $x \in S_h(M)$ ($S_h(M)$ is a set of all self-adjoint operators in $S(M)$) limit $\tau(\{|x| > \lambda\}) \xrightarrow{\lambda \rightarrow \infty} 0$, hence for a sequence of polynomial $P_j(y)$ in \mathbb{R} converging to $\chi_{\{|y| > \lambda\}}(y)$ pointwise, sequence $P_j(x)$ converges to $\chi_{\{|x| > \lambda\}}(x)$ stochastically. Then by [6] Proposition 3.2

$$(33) \quad \begin{aligned} \tau(\{|B_n(x_{e_i})| > \lambda\}) &= \lim_j \tau(P_j(B_n(x_{e_i}))) = \lim_j \tau(P_j(B_n(e_i x_{e_i}))) = \\ &= \lim_j \tau(P_j(e_i B_n(x) e_i)) = \lim_j \tau(e_i P_j(B_n(x))) = \tau(e_i\{|B_n(x)| > \lambda\}). \end{aligned}$$

Statement 31 follows now from the fact that (it follows from B_n commutes with e_i and 3)

$$(34) \quad \begin{aligned} \tau(\{|B_n(x)| > \lambda_1 + \lambda_2\}) &= \tau(\{|(e_1 + e_2)B_n(x)(e_1 + e_2)| > \lambda_1 + \lambda_2\}) = \\ &= \tau(\{|(e_1 B_n(x) e_1 + e_2 B_n(x) e_2)| > \lambda_1 + \lambda_2\}) \leq \\ &= \tau(\{|B_n(x_{e_1})| > \lambda_1\}) + \tau(\{|B_n(x_{e_2})| > \lambda_2\}). \end{aligned}$$

□

Remark 2.1. We are going to use 29 in the next section when dealing with stochastic ergodic theorem, since under the conditions of the stochastic ergodic theorem estimate 29 has place.

3. STOCHASTIC ERGODIC THEOREMS

In this section we establish stochastic convergence of the bounded Besicovitch sequences, and show stochastic ergodic theorems for uniform subsequences.

In this section we use following assumptions: M is a von Neumann algebra with faithful normal tracial state τ , and α is an $*$ -automorphism of algebra M . Denote by $A_n(x) = \frac{1}{n} \sum_{l=1}^{n-1} \alpha^l(x)$, for $x \in M$. Define α' as a linear map on $L_1(M, \tau)$ satisfying $\tau(x \cdot \alpha(y)) = \tau(\alpha'(x)y)$ for $x \in L_1(M, \tau)$, $y \in M$, and $A'_n(x) = \frac{1}{n} \sum_{l=1}^{n-1} \alpha'^l(x)$, for $x \in L_1(M, \tau)$.

Let us recall some definitions from Grabarnik and Katz [9] and Chilin Litvinov and Skalski [2].

Definition 3.1. A positive operator $h \in M_+$ is called **weakly wandering** if

$$(35) \quad \|A_n(h)\|_\infty \xrightarrow{n \rightarrow \infty} 0$$

The following definition is due to Ryll-Nardzewski [15].

Definition 3.2. Let \mathbb{C}_1 denote the unit circle in \mathbb{C} . A **trigonometric polynomial** is a map $P_k(n) : \mathbb{N} \mapsto \mathbb{C}$, where $P_k(n) = \sum_{j=0}^{k-1} b_j \cdot \lambda_j^n$ for $\{\lambda_j\}_{j=0}^{k-1} \subset \mathbb{C}_1$.

Bounded Besicovitch sequences are bounded sequences from the l_1 -average closure of the trigonometric polynomials.

More precisely,

Definition 3.3. A sequence β_n of complex numbers is called a **Bounded Besicovitch sequence** (BB-sequence) if

- (i) $|\beta_n| \leq C < \infty$ for every $n \in \mathbb{N}$ and
- (ii) For every $\epsilon > 0$, there exists a trigonometric polynomial P_k such that

$$(36) \quad \limsup_n \frac{1}{n} \sum_{j=1}^{n-1} |\beta_j - P_k(j)| < \epsilon$$

Let μ be the normalized Lebesgue measure (Radon measure) on \mathbb{C}_1 . Let \tilde{M} be the von Neumann algebra of all essentially bounded ultra-weakly measurable functions $f : (\mathbb{C}_1, \mu) \rightarrow M$. Algebra \tilde{M} is isomorphic to $L_\infty(\mathbb{C}_1, \mu) \bar{\otimes} M$ -which is a W^* tensor product of $L_\infty(\mathbb{C}_1, \mu)$ and M , \tilde{M} is a dual to the space $L_1(\mathbb{C}_1, \mu) \bar{\otimes} M_*$ (for definition of W^* tensor product and form of the predual space of the W^* tensor product see for example Takesaki, [19], Theorem IV.7.17). The space $L_1(\mathbb{C}_1, \mu) \bar{\otimes} M_*$ maybe considered as a set of L_1 functions on (\mathbb{C}_1, μ) with values in M_* . Algebra \tilde{M} has a natural trace $\tilde{\tau}(f) = \int_{\mathbb{C}_1} \tau(f(z)) d\mu(z)$, and \tilde{M}_* is isomorphic to $L_1(\tilde{M}, \tilde{\tau})$.

Let σ be an automorphism of (\mathbb{C}_1, μ) as a Lebesgue space with measure. We define automorphism $\alpha \bar{\otimes} \sigma$ of $(\tilde{M}, \tilde{\tau})$ as a closure of the linear extension of automorphism acting on $(\tilde{M}, \tilde{\tau}) \ni x(z)$ as $\alpha \bar{\otimes} \sigma(x(z)) = \alpha(x(\sigma(z)))$.

Example 3.1. An example of such an automorphism is $\tilde{\alpha}_\lambda(x(z)) = \alpha(x(\lambda \cdot z))$, for $\lambda \in \mathbb{C}_1$.

In this case

$$(37) \quad A_n(x) = \frac{1}{n} \sum_{l=1}^{n-1} \tilde{\alpha}_\lambda^l(x) = \frac{1}{n} \sum_{l=1}^{n-1} \alpha^l(x(\lambda^l \cdot z)).$$

In particular, if $x(z) \equiv z \cdot x$ for $x \in M$ then

$$(38) \quad A_n(x \cdot z) = z \cdot \frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha^l(x).$$

The following lemma connects stochastic convergence in $L_1(\tilde{M}, \tilde{\tau})$ with pointwise convergence on C_1 and stochastic convergence in M (cmp. with [2]).

Lemma 3.2. i) If $L_1(\tilde{M}, \tilde{\tau}) \ni x_n \xrightarrow{n \rightarrow \infty} x_0 \in L_1(\tilde{M}, \tilde{\tau})$ b.a.u. , then $x_n(z) \xrightarrow{n \rightarrow \infty} x_0(z)$ stochastically for almost every $z \in \mathbb{C}_1$
ii) Suppose that h is a weakly wandering operator with support $\text{supp}(h) = \mathbb{I}$ for sequence A_n . Then $A'_n(x)$ converges to 0 stochastically.
iii) Let algebra $\mathcal{N} = (M, \tau) \bar{\otimes} L_\infty(X, \mu)$, (here X is a separable Hausdorff compact set, and μ is Lebesgue measure), α is an automorphism of M , and σ is an automorphism of $L_\infty(X, \mu)$. Then $\alpha \bar{\otimes} \sigma$ is an automorphism of \mathcal{N} . Suppose that h is a weakly wandering operator with support $\text{supp}(h) = \mathbb{I}$ for sequence A_n corresponding to automorphism $\alpha \bar{\otimes} \sigma$. Then $A'_n(x(z))$ converges to 0 stochastically for almost every $z \in \mathbb{C}_1$.

Proof. Part i) follows from [2], Lemma 4.1 which states that under the hypothesis of part i) b.a.u. convergence of $x_n(z)$ to $x_0(z)$ for almost every z in \mathbb{C}_1 , (hence double side stochastic convergence), and the fact that double side stochastic convergence is equivalent to (one sided) stochastic convergence (see [2], Theorem 2.2).

Part ii) We suppose that $x \in L_1(M, \tau)_+$ and $A'_n(x)$ is a sequence satisfying

$$(39) \quad \tau(A'_n(x)h) \rightarrow 0 \text{ for } n \rightarrow \infty.$$

The following inequality is valid:

$$(40) \quad ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) \leq \tau(A'_n(x)h).$$

Indeed, for projections $e_1, e_2 \in P(M)$, we have $e_1 e_2 e_1 \geq e_1 \wedge e_2$. To see that note that since $e_1 \wedge e_2$ commutes with e_1, e_2 , we have $(\mathbb{I} - e_1 \wedge e_2)e_1 e_2 e_1 (e_1 \wedge e_2) = 0$, and, hence $e_1 e_2 e_1 = (\mathbb{I} - e_1 \wedge e_2)e_1 e_2 e_1 (\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2)e_1 e_2 e_1 (e_1 \wedge e_2) = (\mathbb{I} - e_1 \wedge e_2)e_1 e_2 e_1 (\mathbb{I} - e_1 \wedge e_2) + (e_1 \wedge e_2)$.

Then,

$$(41) \quad \begin{aligned} ts \cdot \tau(\{A'_n(x) > t\} \wedge \{h > s\}) &\leq t\tau(\{A'_n(x) > t\}s\{h > s\}\{A'_n(x) > t\}) \leq \\ t\tau(\{A'_n(x) > t\}h\{A'_n(x) > t\}) &= t\tau(\{A'_n(x) > t\}h) \leq \\ \tau(A'_n(x)h). \end{aligned}$$

Hence, 40 is valid.

Furthermore,

$$(42) \quad \tau(\{A'_n(x) > t\}) \leq \frac{1}{ts} \tau(A'_n(x)h) + \tau(\mathbb{I} - \{h > s\}).$$

The latter inequality follows from 40, and the fact that $\tau(e_1) \leq \tau(e_1 \wedge e_2) + \tau(\mathbb{I} - e_2)$. Indeed,

$$(43) \quad \begin{aligned} \tau(e_1 - e_1 \wedge e_2) &= \tau((\mathbb{I} - e_1 \wedge e_2)e_1(\mathbb{I} - e_1 \wedge e_2)) = \tau(e_1(\mathbb{I} - e_1 \wedge e_2)e_1) \leq \\ \tau(e_1(\mathbb{I} - e_2)e_1) &= \tau(e_1(\mathbb{I} - e_2)) \leq \tau(\mathbb{I} - e_2). \end{aligned}$$

Hence 42 is valid.

Note that inequality 42 with the fact that $\tau(A'_n(x)h) \xrightarrow{n \rightarrow \infty} 0$ implies that $\sup_{n \in \mathbb{N}} \tau(\{|A_n(b)| > \lambda \|b\|)\} \leq C(\lambda)$. Indeed, sequence $\{\tau(A'_n(x)h)\}_{n=1}^\infty$ is bounded by constant C_0 as a converging sequence. Choose a monotonically decreasing sequence of $\{s_j\}_{j=1}^\infty \subset \mathbb{R}_+$ such that $\tau(\mathbb{I} - \{h > s_j\}) < 2^{-j}$, and $t_j = 2^j s_j^{-1}$. Then

$$(44) \quad \tau(\{A'_n(x) > t_j\}) \leq \frac{1}{t_j s_j} C_0 + 2^{-j} = (C_0 + 1)2^{-j}.$$

Hence the condition of theorem 2.4 is satisfied. From the theorem follows stochastic convergence of the $A'_n(x)$ since for the dense subset in $L_1(M, \tau)$ of view $x - A'_k(x) + \tilde{x}$ (here $\tilde{x} \in M$ is an α' -invariant element, see [12], Theorem 1.5 (iii) on page 273) for $x \in M \cap L_1(M, \tau)$ convergence is in L_1 , and hence stochastically.

Part iii). The proof follows line of the proof for ii). We provide only necessary modifications. Let E_1 be a conditional expectation with respect to trace $\tau \otimes \mu$ of $(M, \tau) \overline{\otimes} L_\infty(X, \mu)$ onto $(M, \tau) \overline{\otimes} \text{Const}(X, \mu)$, and E_2 be a conditional expectation with respect to trace $\tau \otimes \mu$ of $(M, \tau) \overline{\otimes} L_\infty(X, \mu)$ onto $\mathbb{C} \cdot \mathbb{I} \overline{\otimes} L_\infty(X, \mu)$, (for definition of conditional expectation trace with respect to $\tau \otimes \mu$ and its existence see [19]). Due to the form of the $\alpha \otimes \sigma$, both E_j 's commute with A_n , for $j = 1, 2$.

Since

$$(45) \quad \|A_n(h)\|_\infty \geq \|E_1 A_n(h)\|_\infty = \|A_n(E_1 h)\|_\infty,$$

and $\text{supp}(h) \leq \text{supp}(E_1 h)$ is valid, it follows that $\text{supp}(E_1 h) = \mathbb{I}$. Indeed, $x \geq 0$, $x \neq 0$ implies $\tau(E_1 x) = \tau(x) > 0$ hence $0 < \tau((E_1 a)h) = \tau(a(E_1 h))$ and $\text{supp}(E_1 h) = \mathbb{I}$ for every $a \in M$.

Hence $E_1(h)$ is a weakly wandering operator.

For positive $x(z) \in L_1(M, \tau) \otimes L_1(X, \mu)$ holds

$$(46) \quad \|x\|_1 = \int_X \|x(z)\|_1 \cdot d\mu(z),$$

hence $\|x(z)\|_1$ is an $L_1(X, \mu)$ function. Applying classical Hopf inequality (see for example [12], Theorem 2.1, p. 8) we get

$$(47) \quad \mu(\sup_n \{ \|A'_n(x)(z)\|_1 > \lambda \}) \leq \frac{\text{Const}}{\lambda} \int_X \|x(z)\|_1 \cdot d\mu(z),$$

or, outside of a set $X_0 \subset X$ of small measure the value of $\|A'_n(x)(z)\|_1$ is uniformly bounded. Proceeding like in the part ii) applied for every $z \in X_0$, we get stochastic converges for every $z \in X_0$. \square

Theorem 3.3 (Neveu Decomposition for the special case of tensor product of von Neumann algebras). *Let algebra $\mathcal{N} = (M, \tau) \otimes L_\infty(X, \mu)$, (here X is a Hausdorff separable compact set, and μ is Lebesgue measure), α is an automorphism of M , and σ is an automorphism of $L_\infty(X, \mu)$. Then $\tilde{\alpha} = \alpha \otimes \sigma$ is an automorphism of \mathcal{N} . Suppose that in addition automorphism σ is ergodic. Then there exists an $\tilde{\alpha}$ invariant projection in \mathcal{N} of view $e_1 = e_{11} \otimes \mathbb{I}$, $e_2 = \mathbb{I} - e_1$ with $e_1(z) = e_M$ for almost every $z \in X$ such that*

- i) *There exists a normal state ρ on (N) with $\text{supp}(\rho) = e_1$ and for almost each $z \in X$, $\rho(z)$ is invariant with respect to automorphism α' ;*
- ii) *There exists a weakly wandering operator $h \in \mathcal{N}$ with $\text{supp}(h) = e_2$ and for almost each $z \in X$, $h(z)$ is a weakly wandering operator in M .*

Proof. Corollary 1.1 of [9] implies existence of the projection \tilde{e}_1 in \mathcal{N} such that i) there exists $\tilde{\alpha}'$ invariant normal state ρ with support $\text{supp}(\rho) = \tilde{e}_1$ and ii) there exists a weakly wandering operator $h \in \mathcal{N}$ with support $\mathbb{I} - \tilde{e}_1$. Our goal is to show that similar statements are valid for almost every $z \in X$.

Since σ is ergodic, then for every $x \in M \otimes \text{Const}(X, \mu)$ (constant function on X with values in M) holds

$$(48) \quad \rho(z)(x(z)) = (\tilde{\alpha}'\rho(z))(x(z)) = \rho(z)(\alpha(x(\sigma(z)))) = \rho(z)(\alpha(x(z))) = (\alpha'(\rho(z)))(x(z))$$

,or $\rho(z)$ is α' invariant. Suppose that function $z \rightarrow \rho(z)$ is not constant or $z \rightarrow \rho(z)$ is such that there exists real $r_0 \in \mathbb{R}_+$ and $x(z) \equiv x_0 \in M_+$ with $\mu(\{z \in X | \rho(z)(x(z)) \leq r_0\}) > 0$ and $\mu(\{z \in X | \rho(z)(x(z)) < r_0\}) > 0$. Since σ is ergodic, there exists $n \in \mathbb{N}$ such that

$$(49) \quad \mu(\sigma^{-n}(\{z \in X | \rho(z)(x(z)) \leq r_0\}) \cap \{z \in X | \rho(z)(x(z)) < r_0\}) > 0.$$

Hence,

$$(50) \quad \rho(z)(x(z)) = (\tilde{\alpha}'^n \rho(z))(x(z)) = (\alpha')^n(\rho(z))(x(\sigma^n(z))) = \rho(z)(x(\sigma^n(z))) = \rho(\sigma^{-n}z)(x((z))),$$

or $r_0 \geq \rho(z)(x_0) = \rho(\sigma^{-n}z)(x_0) < r_0$. Contradiction shows that function $z \rightarrow \rho(z)$ is constant.

This implies that $\text{supp}(\rho) = \text{supp}(\rho(z)) = \tilde{e}_1(z)$ is constant.

Part ii) follows directly arguments of proof 45. \square

Theorem 3.4. *Let algebra $\mathcal{N} = (M, \tau) \overline{\otimes} L_\infty(X, \mu)$, (here X is a separable Hausdorff compact set, and μ is normalized Lebesgue measure), α is an automorphism of M , and σ is an automorphism of $L_\infty(X, \mu)$. Then $\tilde{\alpha} = \alpha \otimes \sigma$ is an automorphism of \mathcal{N} . Suppose that in addition automorphism σ is ergodic. Then for almost every $z \in X$ the averages $A'_n(x(z))$ converges stochastically.*

Proof. Proof of the theorem follows directly from 3.3 and 3.2 applied the part where there exists weakly wandering operator, and from the regular individual ergodic theorem [20] applied to the part, where an invariant normal state exists, and Proposition 2.5. \square

Now we are in a position to prove stochastic convergence of the bounded Besicovitch sequences.

Theorem 3.5 (Stochastic Ergodic Theorem for bounded Besicovitch sequences). *Let $\{\beta_j\}_{j=1}^\infty$ be a bounded Besicovitch sequence. Let M be a von Neumann algebra with finite faithful normal tracial state τ . Let α be an automorphism of M . Then the sequence*

$$\tilde{A}_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \beta_j \alpha'^j(x)$$

converges stochastically for $x \in L_1(M, \tau)$.

Proof. Suppose first that bounded Besicovitch sequence $\{\beta_j\}_{j=1}^\infty$ is a trigonometric polynomial $P_k(j)$. Then the statement of the theorem is valid.

Indeed, choosing $\tilde{\alpha}$ as in example 3.1 we get from theorem 2.4 and the fact that irrational rotation on the \mathbb{C}_1 is ergodic (Equidistribution Kronecker-Weyl Theorem, see for ex. [11] p. 146) that

$$(51) \quad A_n(x \cdot z) = z \cdot \frac{1}{n} \sum_{l=1}^{n-1} \lambda'^l \cdot \alpha^l(x),$$

hence

$$\frac{1}{n} \sum_{l=1}^{n-1} \lambda^l \cdot \alpha'^l(x)$$

converges stochastically for irrational λ .

For the rational λ convergence follows from the fact that it is a finite combination of averages of the α'^m , where m is denominator.

Taking linear combinations of terms as in 51 implies the statement for trigonometric polynomials.

Statement of the theorem is valid for the $x \in M \cap S(M)$. Indeed, using approximation of the BB sequence by trigonometric polynomials as in 36 one gets for $A_n(k, x) = \frac{1}{n} \sum_{l=1}^{n-1} P_k(l) \cdot \alpha'^l(x)$

$$(52) \quad \|\tilde{A}_n(x) - A_n(k, x)\|_\infty \leq \frac{1}{n} \left(\sum_{l=0}^{n-1} |\beta_l - P_k(l)| \right) \cdot \|x\|_\infty$$

and, hence, stochastic convergence.

Note also that for every $x \in L_1(M, \tau)$

$$(53) \quad \|\tilde{A}_n(x) - A_n(k, x)\|_1 \leq \frac{1}{n} \left(\sum_{l=0}^{n-1} |\beta_l - P_k(l)| \right) \cdot \|x\|_1.$$

Hence by remark 2.1 averages $\tilde{A}_n(x)$ are uniformly bounded in the sense of 11.

Result of the theorem follows from the Stochastic Banach Principle, Theorem 3.2.4 and density of $M \cap S(M)$ in $L_1(M, \tau)$. \square

The following theorem is implied by the stochastic ergodic theorem for bounded Besicovitch sequences. (cmp. [13])

For the following definitions see for example [12], p. 260.

Let σ be a homeomorphism of a compact metric space X with metric ϱ such that all powers of σ^l are equicontinuous. Assume also that there exists $z \in X$ with dense orbit $\sigma^l(z)$ in X . Then there exists a unique (hence ergodic) σ invariant measure ν on the σ algebra of Borel sets \mathfrak{B} . Each non-empty open set has a positive ν measure.

A sequence u_j is called **uniform** if there exists such dynamical system $(X, \mathfrak{B}, \nu, \sigma)$ and a set $Y \in \mathfrak{B}$ with $\nu(\partial Y) = 0$ and $\nu(Y) > 0$ and point $y \in X$ with $u_j = j^{th}$ entry time of orbit of y into Y .

Theorem 3.6. *Let M, τ, α be as in previous theorem, $\{u_j\}_{j \geq 0}$ be a uniform sequence. Then averages*

$$\frac{1}{n} \sum_{j=0}^{n-1} \alpha'^{u_j} x$$

converge stochastically for $x \in L_1(M, \tau)$.

Proof. Follows from the previous theorem and the fact (see [15]) that any uniform sequence is a bounded Besicovitch sequence. \square

Remark 3.1. Similar results remain valid for the case when M is a semifinite JBW algebra with faithful normal trace τ

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